ON A PROPERTY OF PERIODIC SOLUTIONS OF QUASI-LINEAR AUTONOMOUS SYSTEMS WITH SEVERAL DEGREES OF FREEDOM

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We shall consider a mechanical system with n degrees of freedom whose equation has the form

$$\sum_{k=1}^{n} (a_{ik}\ddot{x}_k + c_{ik}x_k) = \mu F_i(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, \mu) \qquad (i = 1, \ldots, n) \tag{1}$$

The functions $F_i(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, \mu)$ are assumed to be analytic in all their arguments within the regions of their variables. The quantity μ is a small parameter.

The generating system

$$\sum_{k=1}^{n} (a_{ik}\ddot{x}_k + c_{ik}x_k) = 0 \qquad (i = 1, \ldots, n), \qquad a_{ik} = a_{ki}, \qquad c_{ik} = c_{ki} \tag{2}$$

represents a linear conservative system with constant coefficients whose kinetic and potential energies are given by the homogeneous quadratic forms

$$T = \frac{1}{2} \sum_{i,\,k=1}^{n} a_{ik}\dot{x}_i\dot{x}_k, \qquad V = \frac{1}{2} \sum_{i,\,k=1}^{n} c_{ik}x_ix_k$$

It is known that the quadratic form which represents the kinetic energy is positive-definite. Hence

$$\Delta_0 = |a_{ik}| > 0 \tag{3}$$

If one looks for the particular solutions of the system (2) in the form

$$x_{k0}(t) = A_k \cos \omega t + \frac{B_k}{\omega} \sin \omega t$$

then one obtains the following system of linear homogeneous equations for the determination of the A_k:

$$\sum_{k=1}^{n} (c_{ik} - \omega^2 a_{ik}) A_k = 0 \qquad (i = 1, \ldots, n)$$

An analogous system determines the coefficients B_k. The condition for the existence of a nontrivial solution of these systems is given by the equation

$$\Delta (\omega^2) = | c_{ik} - \omega^2 a_{ik} | = 0 \tag{4}$$

This equation of the nth degree in ω^2 has n real roots by Sylvester's theorem. All these roots will be positive if the potential energy is also represented by a positive-definite quadratic form.

Let us suppose that the root ω_r^2 is a simple (not multiple) root. Then we have the following relation for the coefficients A_{kr} and B_{kr} which belong to the oscillation with frequency ω_r:

$$p_k^{(r)} = \frac{A_{kr}}{A_{1r}} = \frac{B_{kr}}{B_{1r}} = \frac{\Delta_{ik} (\omega_r^2)}{\Delta_{i1} (\omega_r^2)} \qquad (i = 1, \ldots, n) \tag{5}$$

where $\Delta_{ik}(\omega_r^2)$ is the algebraic cofactor of the element $c_{ik} - \omega_k^2 a_{ik}$ in the determinant $\Delta(\omega_r^2)$. It is obvious that $p_1^{(r)} = 1$.

Suppose that the solution of the generating system contains l frequencies which are commensurate with each other, and are all distinct. For example, let them be $\omega_1, \omega_2, \ldots, \omega_l$. To these frequencies there corresponds a periodic solution of the system with some period, say T_0. Hereby the intial conditions of the system must be chosen in the following way:

$$x_{k0} (0) = \sum_{r=1}^{l} p_k^{(r)} A_r, \qquad \dot{x}_{k0} (0) = \sum_{r=1}^{l-1} p_k^{(r)} B_r \tag{6}$$

The first subscript 1 in the A_{1r} and B_{1r} has been omitted. Because of the autonomous nature of the system, it is assumed that $B_l = 0$. The quantities $p_k^{(r)}$ are determined by Formulas (5).

The solution of the generating system can be represented in the following form:

$$x_{10} (t) = x_0^{(1)} (t) + x_0^{(2)} (t) + \ldots + x_0^{(l)} (t) \tag{7}$$
$$x_{k0} (t) = p_k^{(1)} x_0^{(1)} (t) + p_k^{(2)} x_0^{(2)} (t) + \ldots + p_k^{(l)} x_0^{(l)} (t) \qquad (k = 2, 3, \ldots, n)$$

The form of the solution for a linear system does not depend on the commensurability of the frequencies $\omega_1, \ldots, \omega_l$.

Let us assume that to the indicated solution of the generating system there corresponds a periodic solution of the system (1), which becomes the generating one when $\mu = 0$. This solution will have the period $T = T_0 + a$, where a is a quantity that vanishes for $\mu = 0$. We shall prove that such a periodic solution can be represented in a form which is entirely analogous to the form (7) of the solution of the generating linear equation.

In accordance with the method of a small parameter, the initial conditions for the system (1) are obtained from the initial conditions of the generating system through the addition of certain quantities which vanish when $\mu = 0$. In the given case it is convenient to take the initial conditions in the form

$$x_k(0) = x_{k0}(0) + \sum_{j=1}^{n} b_{kj}\beta_j^\circ, \qquad \dot{x}_k(0) = \dot{x}_{k0}(0) + \sum_{j=1}^{n} e_{kj}\gamma_j^\circ$$

where b_{kj} and e_{kj} are as yet undetermined coefficients, while β_j° and γ_j° are functions of μ which vanish when $\mu = 0$.

The solution of the system (1) will depend on the parameters β_j°, γ_j° and μ. Let us suppose that the functions $x_k(t)$ can be expanded into series in integer powers of these parameters. Let us determine those terms of these series which depend on the parameters β_j° and γ_j° but not on the parameter μ. Of these terms there will remain only those which are linear in β_j° and γ_j°. The rest of these terms will vanish because their coefficients satisfy the system of homogeneous equations (2) with zero initial conditions. Thus, the functions $x_k(t)$ will have the form

$$x_k(t, \beta_j^\circ, \gamma_j^\circ, \mu) = x_{k0}(t) + \sum_{j=1}^{n} P_{kj}\beta_j^\circ + \sum_{j=1}^{n} Q_{kj}\gamma_j^\circ + \mu[\ldots]$$

For the determination of the coefficients P_{kj} of the parameter β_j° we have the following system of equations:

$$\sum_{k=1}^{n} (a_{ik}\ddot{P}_{kj} + c_{ik}P_{kj}) = 0 \qquad (i = 1, \ldots, n)$$

and an analogous system for the coefficients Q_{kj} of the parameter γ_j°.

For the construction of the periodic solution of the system (1) with period $T = T_0 + a$ one can use only those frequencies which enter into the solution of the generating system. Furthermore, because of the autonomy of the system (1), the frequency ω_l is not used in the determination of the coefficients Q_{kj}. Taking into consideration the initial conditions, we have

$$P_{kj} = \sum_{r=1}^{l} p_k^{(r)} U_j^{(r)} \cos \omega_r t, \qquad Q_{kj} = \sum_{r=1}^{l-1} p_k^{(r)} V_j^{(r)} \sin \omega_r t$$

where $U_j^{(r)}$ and $V_j^{(r)}$ are some constants. The coefficients b_{kj} and e_{kj} are given by

$$b_{kj} = \sum_{r=1}^{l} p_k^{(r)} U_j^{(r)}, \qquad e_{kj} = \sum_{r=1}^{l-1} p_k^{(r)} V_j^{(r)}$$

Let us introduce the new quantities

$$\beta_r = \sum_{j=1}^{n} U_j^{(r)} \beta_j^\circ \quad (r = 1, \ldots, l), \qquad \gamma_r = \sum_{j=1}^{n} V_j^{(r)} \gamma_j^\circ \quad (r = 1, \ldots, l-1)$$

We obtain

$$x_k(t, \beta_r, \gamma_r, \mu) = x_{k0}(t) + \sum_{r=1}^{l} p_k^{(r)} \beta_r \cos \omega_r t + \sum_{r=1}^{l-1} p_k^{(r)} \frac{\gamma_r}{\omega_r} \sin \omega_r t + \sum_{m=1}^{\infty} \left[C_{km}(t) + \frac{\partial C_{km}}{\partial \beta_1} \beta_1 + \ldots + \frac{\partial C_{km}}{\partial \gamma_1} \gamma_1 + \ldots \right] \mu^m$$

Hereby the initial conditions take the form

$$x_k(0) = x_{k0}(0) + \sum_{r=1}^{l} p_k^{(r)} \beta_r, \quad \dot{x}_k(0) = \dot{x}_{k0}(0) + \sum_{r=1}^{l-1} p_k^{(r)} \gamma_r \qquad (k = 1, \ldots, n) \quad (8)$$

It is not difficult to prove, in a manner analogous to that used in [1] and [2], that one can replace differentiation with respect to β_r and γ_r by differentiation with respect to A_r and B_r, respectively. Thus, finally, we obtain

$$x_k(t, \beta_r, \gamma_r, \mu) = \sum_{r=1}^{l} p_k^{(r)} (A_r + \beta_r) \cos \omega_r t + \sum_{r=1}^{l-1} p_k^{(r)} \frac{B_r + \gamma_r}{\omega_r} \sin \omega_r t + \sum_{m=1}^{\infty} \left[C_{km}(t) + \frac{\partial C_{km}}{\partial A_1} \beta_1 + \ldots + \frac{\partial C_{km}}{\partial B_1} \gamma_1 + \ldots \right] \mu^m \qquad (k = 1, \ldots, n) \quad (9)$$

The coefficients $C_{km}(t)$ satisfy the system of equations

$$\sum_{k=1}^{n} (a_{ik} \ddot{C}_{km} + c_{ik} C_{km}) = H_{im}(t) \qquad (i = 1, \ldots, n) \quad (10)$$

under the initial conditions

$$C_{km}(0) = 0, \qquad \dot{C}_{km}(0) = 0$$

The quantities $H_{im}(t)$ are determined by the formula

$$H_{im}(t) = \frac{1}{(m-1)!}\left(\frac{d^{m-1}F_i}{d\mu^{m-1}}\right)_{\mu=0}$$

where

$$H_{i1}(t) = F_i(x_{10}, \ldots, x_{n0}, \dot{x}_{10}, \ldots, \dot{x}_{n0}, 0)$$

Let us indicate the operator of differentiation with respect to time by D. Then the system (10) can be rewritten in the form

$$\sum_{k=1}^{n} (a_{ik}D^2 + c_{ik})\, C_{km} = H_{im}(t) \qquad (i = 1, \ldots, n) \tag{11}$$

From this we obtain

$$C_{km}(t) = \frac{1}{\Delta^*(D^2)} \sum_{i=1}^{n} \Delta_{ik}^*(D^2)\, H_{im}(t)$$

Here $\Delta^*(D^2)$ is the determinant of the system (11), while $\Delta_{ik}^*(D^2)$ is the algebraic cofactor of the element $a_{ik}D^2 + c_{ik}$. One can easily see that

$$\Delta^*(D^2) = \Delta(-D^2) \tag{12}$$

where Δ is the determinant appearing on the left-hand side of Equation (4). Analogous relations hold for the algebraic cofactors of the other elements of these determinants.

From Formulas (3), (4) and (12) it follows that

$$\Delta^*(D^2) = \Delta_0 \prod_{r=1}^{n} (D^2 + \omega_r^2)$$

Bearing in mind that the frequencies $\omega_1, \ldots, \omega_l$ are all distinct, we make use of the following expansion into partial fractions:

$$\frac{\Delta_{ik}^*(D^2)}{\Delta^*(D^2)} = \frac{1}{\Delta_0}\left[\sum_{r=1}^{l} \frac{K_{ik}^{(r)}}{D^2 + \omega_r^2} + \ldots\right], \qquad K_{ik}^{(r)} = \Delta_{ik}(\omega_r^2)\left[\prod_{s \neq r}^{n} (\omega_s^2 - \omega_r^2)\right]^{-1}$$

Since we are looking for a periodic solution of the system (1) with period $T = T_0 + \alpha$, we must retain in the construction of the coefficients $C_{km}(t)$ only those terms which are connected to the frequencies $\omega_1, \ldots, \omega_l$. Therefore

$$C_{km}(t) = \frac{1}{\Delta_0} \sum_{r=1}^{l} \left[\omega_r \prod_{s \neq r}^{n} (\omega_s^2 - \omega_r^2)\right]^{-1} \int_0^t R_{km}^{(r)}(\tau) \sin \omega_r (t - \tau)\, d\tau \tag{13}$$

where

$$R_{km}{}^{(r)}(t) = \sum_{i=1}^{n} \Delta_{ik}(\omega_r{}^2) H_{im}(t) \tag{14}$$

We introduce the notation

$$C_m^{(r)}(t) = \Big[\Delta_0\omega_r \prod_{s\neq r}^{n} (\omega_s{}^2 - \omega_r{}^2)\Big]^{-1} \int_0^t R_{1m}^{(r)}(\tau)\sin\omega_r(t-\tau)\,d\tau \tag{15}$$

Taking into account relation (5), we obtain

$$C_{1m}(t) = \sum_{r=1}^{l} C_m{}^{(r)}(t), \qquad C_{km}(t) = \sum_{r=1}^{l} p_k{}^{(r)} C_m{}^{(r)}(t), \qquad (k = 2, 3, \ldots, n) \tag{16}$$

From Formulas (6), (8) and (16) it follows directly that the solution of the system (1) will have the form

$$\begin{aligned} x_1(t) &= x^{(1)}(t) + x^{(2)}(t) + \ldots + x^{(l)}(t) \\ x_k(t) &= p_k{}^{(1)} x^{(1)}(t) + p_k{}^{(2)} x^{(2)}(t) + \ldots + p_k{}^{(l)} x^{(l)}(t) \qquad (k = 2, 3, \ldots n) \end{aligned} \tag{17}$$

which proves the assertion made above. The quantities $x^{(r)}(t)$ are determined by the formula

$$\begin{aligned} x^{(r)}(t) = (A_r + \beta_r)\cos\omega_r t + \frac{B_r + \gamma_r}{\omega_r}\sin\omega_r t + \\ + \sum_{m=1}^{\infty}\Big[C_m^{(r)}(t) + \frac{\partial C_m{}^{(r)}}{\partial A_1}\beta_1 + \cdots + \frac{\partial C_m{}^{(r)}}{\partial B_1}\gamma_1 + \cdots\Big]\mu^m \qquad (r=1,\ldots,l) \end{aligned} \tag{18}$$

We note that the expression for $x^{(l)}(t)$ will not enter into the term with $\sin\omega_l t$.

In the particular case when a given frequency, for example ω_1, is not commensurate with any of the other frequencies, we obtain

$$x_k(t) = p_k{}^{(1)} x^{(1)}(t) \qquad (k = 2, 3, \ldots, n) \tag{19}$$

Let us formulate the results obtained.

If the generating solution of the quasi-linear autonomous system (1) contains l different commensurate frequencies which determine a periodic solution with some period T_0, then the corresponding periodic solution of the nonlinear system with period $T = T_0 + \alpha$ (α vanishes when $\mu = 0$) which becomes the generating one when $\mu = 0$, will have the form (17) which is analogous to the form of the solution of the generating linear system.

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